Dylan's MATH211 Notes

Week 1

Lecture 1: *Parametric Intro*

Multivariable calculus is about extending the core topics from Calc I and II to cases with many variables. Historically this originated in physics, but now this necessity shows up all over the place. For example, machine learning makes usage of this very frequently.

Basic Notation :

$$[a, b] \leftarrow \text{ all real numbers } x \text{ s.t. } a \leq x \leq b$$

$$(a, b) \leftarrow \text{ all real numbers } x \text{ s.t. } a < x < b$$

$$\mathbb{R} \leftarrow \text{ all real numbers } x$$

$$\mathbb{R}^2 \leftarrow \text{ all ordered pairs of real numbers}$$

$$\mathbb{R}^n \leftarrow \text{ all ordered collections of real numbers}$$

The intuitive idea that this is describing a Cartesian coordinate set in n-dimensional space.

Function Notation :

$$\begin{aligned} f : A &\longrightarrow B \\ x &\longmapsto f(x) \end{aligned}$$

$$f : \text{ Name of the function}$$

$$A : \text{ domain: the set of all possible inputs}$$

$$B : \text{ codomain : The set of available outputs}$$

$$x : \text{ A given input in A}$$

$$f(x) : \text{Whatever f(x) is}$$

Examples :

$$\begin{aligned} \sin : \mathbb{R} &\longrightarrow \mathbb{R} \\ \theta &\longmapsto (\sin(\theta)) \end{aligned}$$

$$\begin{aligned} f : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{x^3+5}{x-2} \end{aligned}$$

$$\begin{aligned} g : \{3,4,5,6,\ldots\} &\longrightarrow \mathbb{R} \\ n &\longmapsto \begin{cases} 0 \text{ if there are no positive whole numbers s.t } a^n + b^n + c^n \\ 1 \text{ if there are} \end{cases} \end{aligned}$$

Our primary focus as a class revolves around functions $\mathbb{R}^n \to \mathbb{R}^m$

Parametric Equations : A helpful way of writing down function $f : \mathbb{R} \to \mathbb{R}^n (n = 2$ for today.)

$$x(t) = \text{ function of } t \leftarrow (\mathbb{R} \to \mathbb{R})$$

$$y(t) = \text{ function of } t \leftarrow (\mathbb{R} \to \mathbb{R})$$

$$t \mapsto f(x)$$

$x(t)$ describes how x changes as we vary t, $y(t)$ does the same for y.

How does this relate to function notation? Graphic $x(t), y(t)$ as we have done is the same as defining a function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (x(t), y(t)) \end{aligned}$$

For another example :

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (\cos(t), \sin(t)) \end{aligned}$$

$$x(t) = \cos(t)$$

$$y(t) = \sin(t)$$

This will just chart out the unit circle, unable to draw these right now because of a dead pen battery.

$$x(t) = e^{-t}\cos(t)$$

$$y(t) = e^{-t}\sin(t)$$

This is almost like a circle, but because of the dampening factor, it will spiral inwards!

Lecture 2: *Parametric Equations*

Vectors :

So why do we need vectors in multi? Vectors will allow us to break larger problems down into smaller, much more manageable pieces. For our purposes a vector will be a point in $\mathbb{R}^n$ where we visualize it as an arrow from the origin. Taking the example of a standard vector :

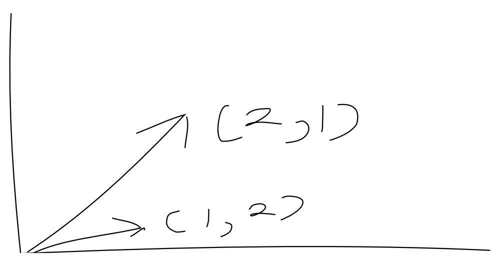

Figure 1: SimpleVectorEx

Warning : Many authors allow vectors that start at points other than the origin. But this doesn't really change anything or do much other than increase complexity.

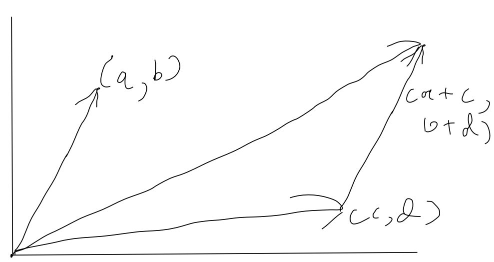

Figure 2: Sept6thAlgebraicVectors

So, to add $\vec{v} + \vec{w}$, we just add the corresponding coordinates :

$$\vec{v} = (v_1, v_2, v_3, \ldots, v_n)$$

$$\vec{w} = (w_1, w_2, w_3, \ldots, w_n)$$

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3, \ldots, v_n + w_n)$$

Consider the following problem :
A wheel is rolling across the x-axis at time $t = 0$, it is resting at the origin. Come up with an parametric equation, $x(t) =?$ and $y(t) = ?$ that describes the position of a point on the whell at time t, which was at the origin at time $t = 0$?

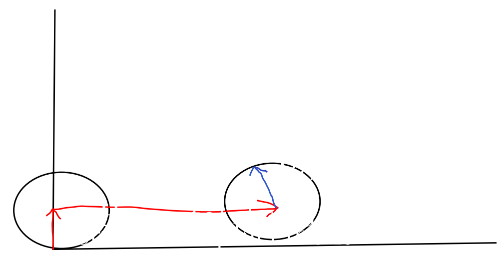

Figure 3: Circle Question

The vertical displacement vector is quite simple, it simply is a $(0, c)$ where c represents the radius of the "ball"

The horizontal displacement vector is also pretty simple, we know the circle will make 1 full revolution once it has traveled $2\pi r$. With our disk of radius 1, we know that the disk will move 2 π units, in 2π time. This means our horizontal vector will be :

$$(t, 0)$$

The spinning vector can be a few things, we know that these two will create a unit circle.

$$x(t) = \cos(t)$$

$$y(t) = \sin(t)$$

However, the ball is rotating in a way such that it's rolling forwards, this means that the terms must be negative, and we have to shift our starting point from being horizontal to facing straight down.

$$x(t) = \cos\left(t - \frac{\pi}{2}\right)$$
$$y(t) = -\sin\left(t - \frac{\pi}{2}\right)$$

Adding everything together :

$$(0,1)+(t,0)+\left(\cos\left(t-\frac{\pi}{2}\right),-\sin\left(t-\frac{\pi}{2}\right)\right)$$

$$\left(t+\cos\left(t-\frac{\pi}{2}\right),1+\sin\left(t-\frac{\pi}{2}\right)\right)$$

Or more readably

$$x(t)=\left(t+\cos\left(t-\frac{\pi}{2}\right)\right)$$

$$y(t)=1+\sin\left(t-\frac{\pi}{2}\right)$$

We have a clock face up on the ground. It is lifted up at a rate of 1 cm/sec The second hand is 10 cm long. Give $x(t)=?$, $y(t)=3$, $z(t)=?$ Describing the tip of the second hand.

Vertical vector :

$$(0,0,t)$$

Rotation vector :

$$x(t)=10\cos(t)$$

$$y(t)=-10\sin(t)$$

These are negative because of the clockwise rotation and the 10 multiplier comes from the length of the second hand. However, since we want a full rotation every 60 seconds, we can then change the rate function :

$$x(t)=10\cos\left(\frac{\pi t}{30}\right)$$

$$y(t)=-10\sin\left(\frac{\pi t}{30}\right)$$

The starting position for the clock then has to be changed, because the clock goes clockwise, we need it to start straight up, meaning we have to send it back $\frac{\pi}{2}$ in it's movement.

$$x(t)=10\cos\left(\frac{\pi t}{30}-\frac{\pi}{2}\right)$$

$$y(t)=-10\sin\left(\frac{\pi t}{30}-\frac{\pi}{2}\right)$$

$$z(t)=t$$

Week 2

Lecture 3: *Special Coordinate Systems*

Last time we saw many examples of rotating systems. Describing these in Cartesian coordinates is sometimes worthwhile but not always. So we seek a potential alternative.

Polar Coordinates

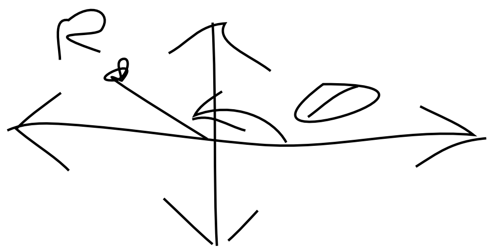

Figure 4: polarExample

We can specify an angle θ from the $+x$ axis and a radius R to specify a point in $\mathbb{R}^2$.

$$\{r, \theta\}$$

You must convert to Cartesian in order to add vectors in this form. This can be annoying but it must be done. Let's figure out how to relate the two to each other.

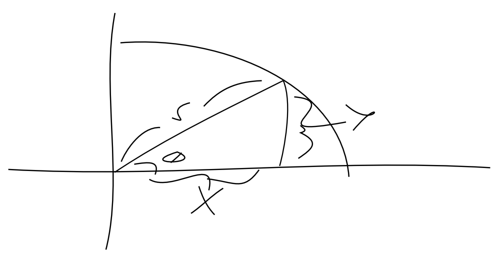

Figure 5: polarToCartesian

Using Pythagorean theorem + trig :

$$x^2 + y^2 = r^2$$

$$\tan\theta = \left(\frac{y}{x}\right)$$

We can use them to turn (x, y) into (r, θ)

$$\sin(\theta) = \frac{y}{r}$$

$$y = r\sin(\theta)$$
$$\cos(\theta) = \frac{x}{r}$$
$$x = r\cos(\theta)$$

Example :
r = 21, $\theta = \frac{\pi}{6}$ compute (x, y)

$$x = 12\cos\left(\frac{\pi}{6}\right) = 12\frac{\sqrt{3}}{2} = 6\sqrt{3}$$

$$y = 12\sin\left(\frac{\pi}{6}\right) = 12\frac{1}{2} = 6$$

Polar Curves A nice way to constant some parametric equation is to start w/ polar and move to Cartesian if needed.

$$\theta(t) = t, r(t) = t$$

Take these as functions of t instead of r.

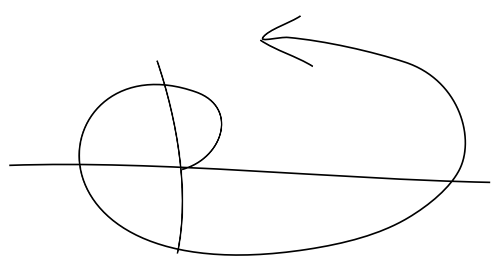

Figure 6: polarCurveExample

To get this in Cartesian coords just substitute in :

$$x(t) = r(t)\cos(\theta(t))$$

$$= t\cos(t)$$

$$y(t) = r(t)\sin(\theta(t))$$

$$= t\sin(t)$$

Example :

$$x(t) = e^{-t}\cos(t)$$

$$y(t) = e^{-t}\sin(t)$$

Can easily be translated to polar coords with :

$$r(t) = e^{-t}, \quad \theta(t) = t$$

As a different example, you can convert form polar coords to Cartesian.

$$r(t) = t, \quad \theta(t) = 2\pi\sin(t)$$

$$x(t) = r(t)\cos(\theta(t)) = t\cos(2\pi\sin(t))$$

$$y(t) = r(t)\sin(\theta(t)) = t\sin(2\pi\sin(t))$$

Very easy to think about in a standard polar system, but it's very confusing in a Cartesian situation.

Cylindrical Coordinates

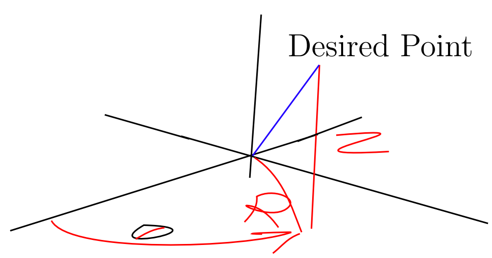

Figure 7: cylindricalCoordinates

$$s = (r, \theta, z)$$

To specify a point in Cylindrical coordinates, give it

1. An angle θ away from the +x-axis in the xy-plane.
2. A radius r away from the origin, in the xy-plane.
3. A z coordinate orthogonal to the xy plane

How to convert form this to Cartesian and back?

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

Cartesian $\longrightarrow$ Cylindrical

$$(x, y, z) \to (r, \theta, z)$$

$$r^2 = x^2 + y^2$$

$$\tan(\theta) = \frac{y}{x}$$

Spherical Coordinates

ϕ is a the angle from z, this angle goes down towards the xy plane. ϕ is always from $0 \to \pi$ and θ goes from $0 \to 2\pi$.

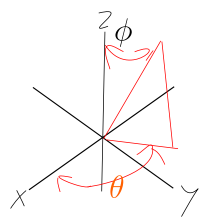

Figure 8: sphericalCoordinates

To specify a point :

1. Angle ϕ measuring from $+z$-axis
2. Angle θ measured in the $xy-$plane away from the $+x$-axis
3. Radius ρ from the origin.

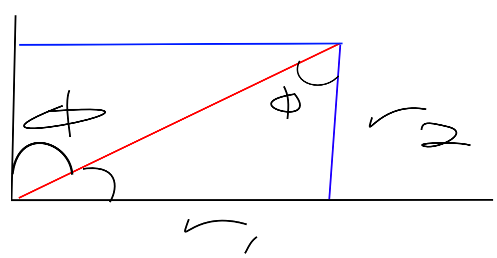

Figure 9: cylindricalTriangle

$$\frac{r_2}{\rho} = \cos(\phi)$$

$$z = r_2 = \rho\cos(\phi)$$

$$\frac{r_1}{\rho} = \sin(\phi)$$

$$r_1 = \rho\sin(\phi)$$

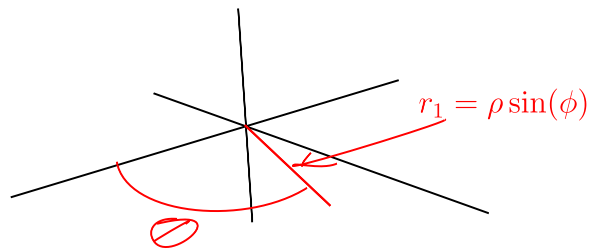

Figure 10: cylindricalTriangle2

Using polar to cartesian

$$x = r_1\cos(\theta)$$

$$= \rho\sin(\phi)coss(\theta)$$

$$y = r_1\sin(\theta)$$

$$= \rho\sin(\phi)\sin(\theta)$$

Combinging together :

$$x = \rho\sin(\phi)\cos(\theta)$$

$$y = \rho\sin(\phi)\sin(\theta)$$

$$z = \rho\cos(\phi)$$

$$\rho = x^2 + y^2 + z^2$$

$$\cos(\phi) = \frac{z}{\rho} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\tan\theta = \frac{y}{x}$$

$$z = \rho\cos(\phi)$$

In cylindrical remember that ϕ is from vertical and θ is from x.

Lecture 4: *Plotting Curves / Surfaces in 2D / 3D*

What options do we have to graph these functions,
Here's one: Given an equation describing a curve / surface, convert the coordinates to something we understand better.

Ex: Graph the polar curve defined by $\theta = \frac{\pi}{4}$

Method 1: Notice that this is just a straight line with slope of 1.
Method 2: Convert to Cartesian

$$\theta = \frac{\pi}{4} \implies \tan\theta = \tan\frac{\pi}{4}$$

$$\frac{y}{x} = 1 \implies y = x$$

Plotting surfaces in 3D

The general idea is that in 2D, adding one constraint gives you a curve. But in 3D, adding one constraint gives you a surface.

Constraining such that $z = 2$ gives you a plane at with contact at $z = 2$ while constraining all points along this plane. Similarly for $x = -1$.

$$x^2 + y^2 + zse = 4 \implies \sqrt{x^2 + y^2 + z^2} = 2$$

This means all points that are distance 2 from $(0, 0, 0)$

Cylindrical

$$r = 2$$

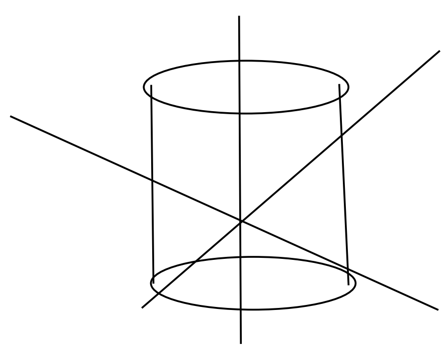

Figure 11: r=2Cylinder

$$r^2 + z^2 = 4$$

When locking just the ϕ value or one of the angles, you are left with a cone. Remember these can be a pair of cones. Specifically for spherical.

Lecture 5: *Linear Algebra Review*

Sketch the surface defined by $z = r$ in cylindrical coordinates :

Observe that θ is free.

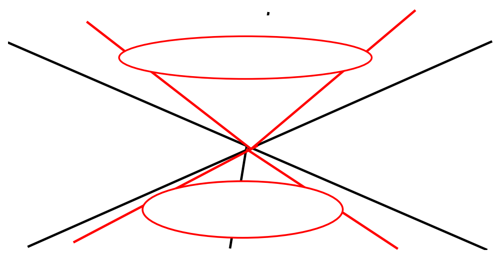

Figure 12: z=r

Vector Arithmetic

We can multiply any vector $\vec{v}$ in $\mathbb{R}$ by a scalar by any real number r.

$$\vec{v} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots \vec{v}_n\}$$

$$r\vec{v} = \{r\vec{v}_1, r\vec{v}_2, r\vec{v}_3, \ldots, r\vec{v}_n\}$$

More simple linear algebra review that is tedious to typeset. . . . .

Dot Product

$$\vec{v} \cdot \vec{w} = (\vec{v}_1\vec{w}_1) + (\vec{v}_2\vec{w}_2) + \ldots + (\vec{v}_n\vec{w}_n)$$

This is the square of the length of a vector :

$$(x, y, z) \cdot (x, y, z) = x^2 + y^2 + z^2$$

Or more shortly :

$$\vec{v} \cdot \vec{v} = ||\vec{v}||^2$$

Theorem: suppose $\vec{v}, \vec{w}$ are non-zero vectors in $\mathbb{R}^n$ Let θ be the angle between them. Then $\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}|| \cos(\theta)$ The dot product is so convenient that it's frequently very nice to define things in terms of it.

1. We say that $\vec{v}$ in $\mathbb{R}^2$ is a unit vector if $\vec{v} \cdot \vec{v} = 1$
2. We say that $\vec{v}, \vec{w}$ in $\mathbb{R}^n$ are orthogonal if $\vec{v} \cdot \vec{w} = 0$
3. Planes can be defined as all of the vectors orthogonal to a given $\vec{v}$ in $\mathbb{R}^2$.

Week 3

Lecture 6: *Planes and Normal Vectors*

If c is a real number, then

$$c\vec{v} \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$$

Let $\vec{v}$ be the normal vector to a plane in $\mathbb{R}^3$. We can define this plane as a plane of points orthogonal to $\vec{v}$.

This is very useful to give an easy way to describe the angle at which something will bounce off a surface. It can be seen as a mirroring of the angle from a normal vector. To compute the angle we can use a dot product, but this is easiest when the vectors have length 1.

Normalization :

Q : Given a vector $\vec{v}$ in $\mathbb{R}^n$, how can we find a vector $\vec{w}$ in the same direction?
A : We want a real $c > 0$ s.t $||c\vec{v}|| \implies 1||c\vec{v}||^2 = 1 \implies (c\vec{v}) \cdot (c\vec{v}) = 1$

$$c^2\vec{v} \cdot \vec{v} = c^2||\vec{v}||^2 = 1 \to c = \frac{1}{||\vec{v}||}$$

This means that :

$$\text{Normalized Vector is } \frac{\vec{v}}{\sqrt{\vec{v} \cdot \vec{v}}}$$

WE TALK ABOUT FAST INVERSE SQUARE ROOT IN MULTIVARIABLE LMAOOOOOOOOOO THIS IS ADORABLE!!!

Cross Products :

Cross products allow you to multipl two vectors and get another vector. This is only possible for vectors in $\mathbb{R}^3$. THis is largely due to the fact that polynomials all have a complex root, but this isn't necessarily true for other dimensions.

We take a convention :

$$(1, 0, 0) = \hat{i}$$

$$(0,1,0) = \hat{j}$$

$$(0,0,1) = \hat{k}$$

Definition :

$$\text{Let } \vec{v} = (x,y,z)$$

$$\vec{w} = (a,b,c)$$

$$\vec{v} \times \vec{w} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ z & b & c \end{pmatrix}$$

You have to find the determinant of this matrix :

1. Multiply entries that are diagonally down
2. Multiply entries that are diagonally left and down
3. Add everything in 1 and subtract everything

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{i} = -\hat{k}$$

The cross product of parallel vectors is zero.

There are a few details to note :

1.) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

2.) $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0 \quad \& \quad \vec{w} \cdot (\vec{v} \times \vec{w})$

3.) $||\vec{v} \times \vec{w}|| =$ area of the parallelogram w/ $\vec{v}$ & $\vec{w}$ as sides

$$||\vec{v} \times \vec{w}|| = ||\vec{v}|| \, ||\vec{w}|| \sin(\theta)$$

Following right hand rule convention, two of your fingers are the vectors and the third is the cross product, which makes sense as this third finger will be orthogonal.

Lecture 7: *Vectors Perpendicular to Planes*

$$(1,0,-1) \times (0,2,1)$$

$$(2,-1,2)$$

Compute the area of the triangle with vertices (0,0,0), (1,0,-1), (0,2,1)

We know that the area of a parallelogram is defined by

$$||\vec{v} \times \vec{w}|| = ||\vec{v}||\,||\vec{w}||\sin(\theta)$$

Since we want a triangle we know that it must be half the area of this parallelogram.

$$\frac{1}{2}||(0,2,1) \times (1,0,-1)|| \to \frac{1}{2}3 = \frac{3}{2}$$

Consider the plane that passes through (0,0,0),(1,0,-1), and (0,2,1). Find $\mathbb{R}$ c_1, c_2, c_3 such taht (x, y, z) is a point on this plane if and only if $c_1x + c_2y + c_3z = 0$.

We can see that $c_1x + c_2y + c_3z = 0$. This can be expanded to : $(c_1, c_2, c_3) \cdot (x, y, z) = 0$. This is being a normal vector to the plane.

$$(c_1, c_2, c_3) = (1, 0, -1) \times (0, 2, 1)$$

And since we already solved this :

$$(2, -1, 3)$$

Work equation :

$$||\vec{F}|| \cdot ||\vec{x}|| \text{ or } ||\vec{F}||\;||\vec{X}||\cos(\theta)$$

Flux Equation :

$$\phi = \vec{I} \cdot *\vec{n}$$

Those above are applications of the dot product, but there are also very important usages of the cross product.

Torque :

$$\vec{F} \times \vec{r}$$

This means that when pushing on an object, the direction of it's rotation is cross the direction of the force and the radius. This means that it's rotation will be around a 3rd axis.

The poynting vector :
The oscillation of dipoles forms this vector field. This vector field is used extensively in FM and radio to communicate information. This field is made up of the $\vec{E}$ and $\vec{B}$. This

is the up down, in out oscillation of the $\vec{E}$ and $\vec{B}$ that we see in electromagnetic waves.

$$\vec{E} \times \vec{B} = \text{ poynting vector}$$

Moving electric charges :
Electrons experience a force perpendicular to their direction of movement as well as the overall direction of the magnetic field.

$$\vec{v} \times \vec{B} = \vec{F}$$

Properties of the dot and cross product :

The dot product is commutative :

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

Scalar multiplication commutes :

$$(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$$

Addition distributes :

$$\vec{v} \cdot (\vec{w} + \vec{u}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{u}$$

However the cross product is a bit different:
The cross product is anticommunative :

$$\vec{v} \times \vec{w} = -\vec{w}\vec{v}$$

Scalar multiplication commutes :

$$(c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})$$

Addition distributes :

$$\vec{v} \times (\vec{w} + \vec{w}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{u}$$

Lecture 8: *Limits and Intro Derivatives*

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}^2$
Suppose we want to define $\lim_{t \to t_0} f(t)$
What is the right definition?

Ideal #1 : We could define a way to measure how far $f(t)$ is from $f(t_0)$. Euclidean distance could work. Delta-Epsilon but with vectors

Ideal #2 : Instead of Delta-Epsilon, we will note that vectors in R^n only get closer together iff their components get closer together.

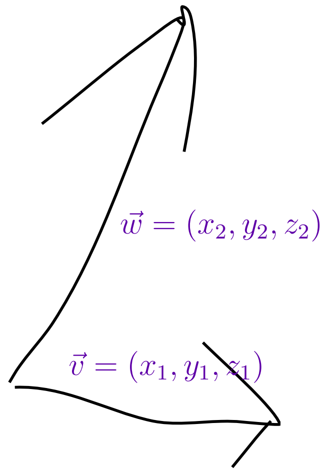

Figure 13: Idea2

$$\vec{v} \approx \vec{w}$$

$$x_1 \approx x_2$$

$$y_1 \approx y_2$$

$$z_1 \approx z_2$$

DEF :

Suppose we have a function

$$f : \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$t \longmapsto (f_1(t), f_2(t), \ldots, f_n(t))$$

We can say that $\lim_{t \to t_0} = (x_1, x_2, x_3, \ldots, x_n)$ if and only if $\lim_{t \to t_0} f_1(t) = x_1$ **EX :**

$$\lim_{t \to \infty} e^{-t}(\cos(t), \sin(t))$$

$$\lim_{t \to \infty} e^{-t} \cos(t), \lim_{t \to \infty} e^{-t} \sin(t)$$
$$(0, 0)$$

EX :

$$\lim_{t\to 0^+}(\frac{1}{t}, t)$$

$$\lim_{t\to 0^+}\frac{1}{t}, \lim_{t\to 0^+} t$$

$$(\text{ UNDE }, 0)$$

Does not exist!

EX:

$$\lim_{t\to 1}\left(\frac{t^2-1}{t-1}, t, t^2\right)$$

$$\lim_{n\to 1} 2t, \lim_{t\to 1} t, \lim_{t\to 1} t^2$$

$$(2, 1, 1)$$

If even one component of the vector doesn't exist, that means the whole vector doesn't exist, everything must converge to a specific point or no dice. We say that $f : \mathbb{R} \to \mathbb{R}^n$ is continuous at t_0 if and only if

$$\lim_{t\to t_0} f(t) = f(t_0)$$

Derivatives :

Suppose that we want to find a tangent line / vector to a parametric curve coming from a function $f : \mathbb{R} \to \mathbb{R}^n$

We can define the limit for a parametric equation as :

$$\lim_{\Delta t\to 0}\frac{f(t_0+\Delta t)-f(t_0)}{\Delta t}$$

Which can intuitively be brought to :

$$(x'(t_0), y'(t_0), z'(t_0))$$

Which will give the derivative for f in terms of the component directions which is called a gradient.

Def :

Let $f : \mathbb{R} \to \mathbb{R}^n$ be a continuous function. We say it is differentiable at t, if its derivative exits at that point :

$$f'(t_0) = \lim_{\Delta t\to 0}\frac{f(t+t_0)-f(t_0)}{\Delta t}$$

Ex:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (1, t, t^2) \end{aligned}$$

$$\begin{aligned} f'(6) &= (\frac{d}{dt}1, \frac{d}{dt}t, \frac{d}{dt}t^2) \\ &= (0, 1, 12) \end{aligned}$$

Week 4

Lecture 9: *Gradient*

Q: What does the derivative of a function $f : \mathbb{R} \to \mathbb{R}^n$ represent?
A1:

$$\begin{aligned} f'(t) &= \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} \end{aligned}$$

So $f'(t)$ is the instantaneous rate of change.
A2:

$$f'(t) = (f_1'(t), f_2'(t), \ldots, f_n'(t))$$

This is the rate of change for all the components, also called the gradient.
A3:

1. Velocity :
   If $f : \mathbb{R} \to \mathbb{R}^n$ represents the position of an object, then $f'(t)$ is the velocity of said object.

   (a) Direction : $f'(t)$ points in the direction of travel.

   (b) Magnitude :

$$\begin{aligned} ||f'(t)|| &= ||\lim_{t \to 0} \frac{\Delta f}{\Delta t}|| \\ &= \lim_{\Delta t \to 0} ||\frac{\Delta f}{\Delta t}|| \\ &= \lim_{\Delta t \to 0} \frac{||\Delta f||}{\Delta t} \end{aligned}$$

Since Δf is the change i position, $||\Delta f||$ is the distance traveled. Therefor, $\lim_{\Delta t \to 0} = \frac{||\Delta f||}{\Delta t}$, speed at a point!

2. Acceleration :
   Suppose that $v : \mathbb{R} \to \mathbb{R}^n$ is the velocity of an object. What is $v'(t)$.

$$v'(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

3. Population:

$$\begin{aligned} P : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (J(t), A(t), T(t)) \end{aligned}$$

   $P(t)$ is a population vector. It tracks how many, jackals, antelopes, and turtles there are at a given time.

$$P'(t) = \text{ vector of population growth / decay}$$

4. Pricing Data

$$\begin{aligned} F : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ c &\longmapsto (\omega, p, t) \end{aligned}$$

   Here's an example where you have the input of copper, and a few wire, pipe, and tuba costs.

$$F'(c) = \text{ rate of change in copper costs as a function of item costs.}$$

Q: What benefit do we get from derivatives :

- Linear Approximation, it's quite easy to use linear congruent lines to make estimations for where larger functions will later go.

Linear Apprxomiation at Time t

$$f(t + \Delta t) \approx f(t) + f'(t)\Delta t$$

EX:

$$
\begin{aligned}
f : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\
t &\longmapsto (t, t^2, t^3)
\end{aligned}
$$

What is the linear approximation at time $t = 1$?

$$f(1) = (1, 1, 1), f'(t) = (1, 2t, 3t^2), f'(1) = (1, 2, 3)$$

$$
\begin{aligned}
f(1 + \Delta t) &\approx f(1) + f'(1)\Delta t \\
&= (1, 1, 1) + (1, 2, 3)\Delta t \\
&= (1 + \Delta t, 1 + 2\Delta t, 1 + 3\Delta t)
\end{aligned}
$$

This function gives a tangent line to the graph of $f(t)$ at $t = t_0$. In other words, the linear approximation is just the tangent line.

- Arc length, Suppose I have a curve $f : [a, b] \to \mathbb{R}^n$. I want to know how long it is. How do I do that?
  After a lot of derivation we get to this formula for an arc length:

  $$\int_a^b ||f'(t)||\, dt$$

  Ex:

  $$f(t) = \cos(t), \sin(t)$$

  FInd the arc length from $t = 0$ to $t = 2\pi$

  $$f'(t) = (-\sin(t), \cos(t))$$

  $$||f'(t)|| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$$

  $$\int_0^{2\pi} 1\, dt = 2\pi$$

Lecture 10: *Product Rule*

A few things to think about :

Thrm: Let $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^n$ be differentiable. Then asd

a

sd

ds

sd

sd

sd

sd

d

s

1. $$\frac{d}{dt}(f(t) \times g(t))$$
$$f'(t) \times g(t) + f(t) \times g'(t)$$

2. $$\frac{d}{dt}(f(t) \cdot g(t))$$
$$f'(t) \cdot g(t) + f(t) \cdot g(t)$$

Yay